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Chaos control and synchronization of unified chaotic systems via linear control

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Abstract

Chaos control and synchronization in the unified chaotic systems is discussed in this paper. Based on the stability theory of a cascade-connected system, control laws are presented to achieve chaos control and synchronization, respectively. The advantage of the proposed controllers is that they are linear and have lower dimensions than that of the states. Simulation results for Lorenz, Lü and Chen chaotic systems are provided to illustrate the effectiveness of the proposed scheme.

1. Introduction

Chaos, as a very interesting nonlinear phenomenon, has been intensively investigated in many fields of science and technology over the last four decades [1–4]. Recently, chaos synchronization has attracted increasing attention from various communities due to its powerfully potential applications in laser physics, chemical reactor, secure communication, biomedical and so on [2–4]. Many methods have been proposed to achieve chaos control and synchronization, such as the passive control method [5], backstepping design method [6], impulsive control method [7], adaptive control method [8], sliding mode control [9,10], control Lyapunov function (CLF) method [11] and nonlinear feedback method [12], etc. The controllers derived from the above methods are nonlinear. In a real industry process, because the linear feedback controllers are economic and easy to implement, they possess a high value in applications. Chaos synchronization via a linear controller was investigated in Refs. [13–15]. Jiang and Zheng [13] treated the problem of chaos synchronization as a special case of observer design. The controller design contains the Lipschitz constants. However, even if the Lipschitz constants are known, the large Lipschitz constants always result in a high gain controller that is not easy to realize in practice. Liu [14] gave a linear controller on the assumption that the nonlinear function of the chaotic system satisfied an upper triangle form (see assumptions in Ref. [14]). Consequently, it is valuable to present a new linear controller for chaotic systems.

To bridge the gap between the Lorenz attractor and the Chen attractor, Lü et al. presented a unified chaotic systems [16]. It presents the Lorenz and Chen systems as two extremes, respectively, and the Lü system as a

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transition system. Since the unified model has been established, several papers have dealt with the chaos control and synchronization of unified chaotic systems. In Ref. [17], Yan et al. applied the sliding mode method to achieve the target. In Ref. [18], Chiang et al. proposed anti-synchronization of uncertain unified chaotic systems with dead-zone nonlinearity. These controllers are nonlinear. In Ref. [15], Wang et al. gave a linear feedback to realize synchronization of the unified chaotic systems. Based on the stability theory of the cascade-connected system [20–23], we propose linear control for chaos control and synchronization for unified chaotic systems. It seems that the controller is simple and the controller gains are less that those given in Ref. [15]. When $0 \leqslant \alpha < \frac{1}{29}$, only one linear controller is required to realize chaos control and synchronization for the unified chaotic systems. When $\frac{1}{29} \leqslant \alpha < 1$, two simple and linear feedback controllers are designed to achieve our target.

Throughout this paper, R^n denotes the n-dimensional Euclidean space. $\|\cdot\|$ denotes either the Euclidean vector norm or the induced matrix spectral norm.

2. Main results

2.1. Preliminaries

The nonlinear differential equations that describe the unified chaotic systems are modeled by

$$\begin{cases} \dot{x}_1 = (25\alpha + 10)(x_2 - x_1), \\ \dot{x}_2 = (28 - 35\alpha)x_1 - x_1 x_3 + (29\alpha - 1)x_2, \\ \dot{x}_3 = x_1 x_2 - (8 + \alpha)x_3/3, \end{cases} \tag{1}$$

where x_1, x_2, and x_3 are state variables and $\alpha \in [0, 1]$ is the system parameter. When $0 \leqslant \alpha < 0.8$, system (1) is called the generalized Lorenz chaotic system. When $\alpha = 0.8$, system (1) is called the Lü chaotic system. When $0.8 < \alpha \leqslant 1$, system (1) is called the generalized Chen chaotic system. The goal of this paper is to design linear controllers for unified chaotic systems (1) to realize chaos control and synchronization, respectively.

For further discussion, two useful lemmas are presented. Lemma 2.1 can be found in Ref. [19] (Theorem 1) or [20]. Lemma 2.2 can be seen in Ref. [21] (Theorem 4.2.10) or in Ref. [22] (Lemma 1). Consider the cascade-connected system described by

$$\begin{cases} \dot{x} = f(x, z), \\ \dot{z} = g(z), \end{cases} \tag{2}$$

where $x \in R^n, z \in R^m, f(0,0) = 0$ and $g(0) = 0$; $f(x,z)$ and $g(z)$ are both C^1 vector fields.

Lemma 2.1 (*Sundarapandian [19] and Feng and Zhang [20]*). *If the system $\dot{x} = f(x, 0)$ and $\dot{z} = g(z)$ are globally asymptotically stable at $x = 0$ and $z = 0$, respectively, and all the trajectories $(x(t), z(t))$ of system* (2) *are bounded, then system* (2) *is globally asymptotically stable at the equilibrium $(x, z) = (0, 0)$.*

Lemma 2.2 (*Burton [21], Jiang et al. [23] and Mei et al. [23]*). *Consider the nonlinear time-varying system $\dot{x} = f(x, t)$, where $x \in U \subset R^n$, $t \in J \subset R^+ = [0, \infty]$. If there exists a differential function $V(x, t) : U \times J \to R$ satisfying the following conditions*:

(1) *There exist a positive constant λ_0 and a scalar function $\bar{\lambda} : U \to R$ such that*

$$\lambda_0 \|x\|^2 \leqslant V(x,t) \leqslant \bar{\lambda}(x)\|x\|^2, \quad \forall (x,t) \in U \times J. \tag{3}$$

(2) *There exist some positive constants $\lambda_V > 0$ and $\varepsilon \geqslant 0$ such that*

$$\dot{V}(x,t)|_{\dot{x}=f(x,t)} \leqslant -\lambda_V \bar{\lambda}(x)\|x\|^2 + \varepsilon, \quad \forall (x,t) \in U \times J, \tag{4}$$

then the solution of the system $\dot{x} = f(x,t)$ *is bounded by*

$$\|x(t;x_0,t_0)\| \leqslant \frac{1}{\lambda_0} V(x_0,t_0)\mathrm{e}^{-\lambda_V(t-t_0)} + \frac{\varepsilon}{\lambda_0\lambda_V}(1-\mathrm{e}^{-\lambda_V(t-t_0)}) \leqslant M, \tag{5}$$

where $M = (1/\lambda_0)V(x_0,t_0) + (\varepsilon/\lambda_0\lambda_V)$.

2.2. Chaos control

In some cases, a chaotic effect is undesirable in practice and it restricts the operating range of many electronic and mechanical devices. Recently, chaos control has attracted a great deal of attention in the engineering society [6,12]. Chaos control means to design a controller that is able to mitigate or eliminate the chaos behavior of nonlinear systems that experience chaotic phenomenon. In this section, a linear controller is presented to globally stabilize the equilibrium point $E = (0,0,0)$ of the unified chaotic systems (1). We assume that the controlled unified chaotic systems are given by

$$\begin{cases} \dot{x}_1 = (25\alpha + 10)(x_2 - x_1) + u_1, \\ \dot{x}_2 = (28 - 35\alpha)x_1 - x_1x_3 + (29\alpha - 1)x_2 + u_2, \\ \dot{x}_3 = x_1x_2 - (8+\alpha)x_3/3. \end{cases} \tag{6}$$

The procedure of controller design consists of 2 steps.

Step 1: Let $u_1 = -(25\alpha + 10)x_2$. Then the first equation of Eq. (6) becomes

$$\dot{x}_1 = -(25\alpha + 10)x_1. \tag{7}$$

Obviously, it is globally asymptotically stable at $x_1 = 0$.

Step 2: Consider the remaining subsystem of system (6), i.e.,

$$\begin{cases} \dot{x}_2 = (28 - 35\alpha)x_1 - x_1x_3 + (29\alpha - 1)x_2 + u_2, \\ \dot{x}_3 = x_1x_2 - (8+\alpha)x_3/3. \end{cases} \tag{8}$$

Choose $u_2 = -Lx_2$ and substitute $x_1 = 0$ into subsystem (8). From the linear system theory, if $L > (29\alpha - 1)$ (for example, we can take $L = 29$) system (8) with $x_1 = 0$ is globally asymptotically stable at $x_2 = x_3 = 0$.

We now verify that the solution of the closed system (8) is bounded with the control $u_2 = -Lx_2(L > (29\alpha - 1))$. Consider the following candidate Lyapunov function:

$$V = \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_3^2. \tag{9}$$

Calculating the derivative of V along the solution of system (8), we obtain

$$\begin{aligned} \dot{V} &= x_2[(28 - 35\alpha)x_1 - x_1x_3 + (29\alpha - 1)x_2 - Lx_2] + x_3\left[x_1x_2 - \frac{8+\alpha}{3}x_3\right] \\ &= -[L - (29\alpha - 1)]x_2^2 + (28 - 35\alpha)x_1x_2 - \frac{8+\alpha}{3}x_3^2 \\ &\leqslant -[L - (29\alpha - 1)]x_2^2 + \gamma_1x_2^2 + \frac{1}{\gamma_1}(28 - 35\alpha)^2x_1^2 - \frac{8+\alpha}{3}x_3^2 \\ &= -[L - (29\alpha - 1) - \gamma_1]x_2^2 - \frac{8+\alpha}{3}x_3^2 + \Gamma, \end{aligned} \tag{10}$$

where $\Gamma = (1/\gamma_1)(28 - 35\alpha)^2x_1^2$ and γ_1 is a positive constant that can be selected arbitrarily. Notice that $|x_1(t,t_0)| < |x_1(0)|$, and so Γ must be bounded. From Lemma 2 and inequality (10), if $L > (29\alpha - 1) + \gamma_1$, then each solution of subsystem (8) is bounded.

Then on addition with Lemma 1 the unified chaotic systems (6) are globally asymptotically stable at the equilibrium point $E = (0,0,0)$.

Remark 2.1. When $0 \leqslant \alpha < \frac{1}{29}$, from inequality (10), the feedback $u_1 = -(25\alpha + 10)\, x_2, u_2 = 0$ is sufficient to globally asymptotically stabilize the system to the origin $E = (0, 0, 0)$.

Remark 2.2. From the above proof we have seen that only one state x_2 is adopted for the feedback.

Remark 2.3. When $\alpha = 0$, the unified chaotic systems become the Lorenz chaotic system. From Remark 2.1, only one linear controller u_1 can globally stabilize the Lorenz chaotic system. This is simpler than M.T. Yassen's nonlinear controller $u = x_1(x_3 - (a + c))$ [6].

2.3. Synchronization of the unified chaotic systems

In general, the two dynamic systems in synchronization are called the master system and the slave system, respectively. This subsection will design a controller to make the trajectories of the slave system asymptotically track the trajectories of the master system (1) , i.e., synchronous. In the following, the master chaotic system is given by Eq. (1) and its slave system is given by

$$\begin{cases} \dot{y}_1 = (25\alpha + 10)(y_2 - y_1) + u_1, \\ \dot{y}_2 = (28 - 35\alpha)y_1 - y_1 y_3 + (29\alpha - 1)y_2 + u_2, \\ \dot{y}_3 = y_1 y_2 - (8 + \alpha)y_3/3. \end{cases} \tag{11}$$

Denote by the synchronization error $e = y - x$. Our aim is to design a controller $u(t) = (u_1, u_2)^{\mathrm{T}}$ such that the controlled system (11) asymptotically synchronizes the master system (1) in the sense that

$$\lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \|y(t, y_0) - x(t, x_0)\| = 0.$$

Subtracting Eq. (1) from Eq. (11) gives the following error system:

$$\begin{cases} \dot{e}_1 = (25\alpha + 10)(e_2 - e_1) + u_1, \\ \dot{e}_2 = (28 - 35\alpha)e_1 - y_1 y_3 + x_1 x_3 + (29\alpha - 1)e_2 + u_2, \\ \dot{e}_3 = y_1 y_2 - x_1 x_2 - (8 + \alpha)e_3/3. \end{cases} \tag{12}$$

From the fact that

$$\begin{cases} x_1 x_3 - y_1 y_3 = -e_1 e_3 - e_1 x_3 - e_3 x_1, \\ y_1 y_2 - x_1 x_2 = e_1 e_2 + e_1 x_2 + e_2 x_1, \end{cases} \tag{13}$$

system (12) can be rewritten in the following form:

$$\begin{cases} \dot{e}_1 = (25\alpha + 10)(e_2 - e_1) + u_1, \\ \dot{e}_2 = (28 - 35\alpha)e_1 - e_1 e_3 - e_1 x_3 - e_3 x_1 + (29\alpha - 1)e_2 + u_2, \\ \dot{e}_3 = e_1 e_2 + e_1 x_2 + e_2 x_1 - (8 + \alpha)e_3/3. \end{cases} \tag{14}$$

Here, we still take two steps to design a linear controller to globally asymptotically stabilize the error system (14).

Step 1: Let $u_1 = -(25\alpha + 10)e_2$ and the first subsystem of (14) becomes

$$\dot{e}_1 = -(25\alpha + 10)e_1. \tag{15}$$

Obviously, for each $\alpha \in [0, 1]$ it is globally asymptotically stable at $e_1 = 0$.

Step 2: Consider the remaining subsystem of the error system (14)

$$\begin{cases} \dot{e}_2 = (28 - 35\alpha)e_1 - e_1 e_3 - e_1 x_3 - e_3 x_1 + (29\alpha - 1)e_2 + u_2, \\ \dot{e}_3 = e_1 e_2 + e_1 x_2 + e_2 x_1 - (8 + \alpha)e_3/3. \end{cases} \tag{16}$$

Substitute $e_1 = 0$, $u_2 = -ke_2(k > 29\alpha - 1)$ into the above system; then, we can have

$$\begin{cases} \dot{e}_2 = -e_3 x_1 + (29\alpha - 1)e_2 - ke_2, \\ \dot{e}_3 = e_2 x_1 - (8+\alpha)e_3/3. \end{cases} \tag{17}$$

For system (17) consider the following candidate Lyapunov function:

$$V = \tfrac{1}{2}e_2^2 + \tfrac{1}{2}e_3^2. \tag{18}$$

The derivative along the solution of system (17) is

$$\begin{aligned} \dot{V} &= -e_2[k - (29\alpha - 1) + x_1 e_3] + e_3(e_2 x_1 - (8+\alpha)e_3/3) \\ &= -[k - (29\alpha - 1)]e_2^2 - \frac{8+\alpha}{3}e_3^2. \end{aligned} \tag{19}$$

From the Lyapunov stability theory we can conclude that subsystem (17) is asymptotically stable at the origin $e_2 = 0, e_3 = 0$.

Now consider the Lyapunov function $V = \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2$. Its derivative along subsystem (16) is

$$\begin{aligned} \dot{V} &= e_2[(28 - 35\alpha)e_1 - e_1e_3 - e_1x_3 - e_3x_1 + (29\alpha - 1)e_2 - ke_2] \\ &\quad + e_3\left[e_1e_2 + e_1x_2 + e_2x_1 - \frac{8+\alpha}{3}e_3\right] \\ &= -[k - (29\alpha - 1)]e_2^2 + (28 - 35\alpha)e_1e_2 - e_1e_2x_3 + e_3e_1x_2 - \frac{8+\alpha}{3}e_3^2 \\ &\leqslant -[k - (29\alpha - 1) - \delta_1 - \delta_2]e_2^2 - \left[\frac{8+\alpha}{3} - \delta_3\right]e_3^2 + \Pi \end{aligned} \tag{20}$$

where

$$\Pi = \frac{1}{\delta_1}(28 - 35\alpha)^2 e_1^2 + \frac{1}{\delta_2}e_1^2x_3^2 + \frac{1}{\delta_3}e_1^2x_2^2.$$

Because the master system (1) is chaotic we know that its states x_1, x_2, and x_3 are bounded. Moreover, from system (15) we can obtain $|e_1(t, t_0)| < |e_1(0)|$, and so Π is bounded. Because δ_1, δ_2 and δ_3 can be arbitrarily selected we can chose them as small as possible such that $k > (29\alpha - 1) + \delta_1 + \delta_2$. Hence, the solution of system (16) is bounded from Lemma 2. Thus, together with Lemma 1 the error system (14) is globally asymptotically stable at $E(0,0,0)$. This means the slave system (11) can asymptotically synchronize the master system (1) with a simple linear feedback controller.

Corollary 2.1. *When* $0 \leqslant \alpha < \frac{1}{29}$ *from the proof process we know that there is only one simple controller* $u_1 = -(25\alpha + 10)e_2 = -(25\alpha + 10)(y_2 - x_2), u_2 = 0$, *which can realize the synchronization of two identical unified chaotic systems.*

Remark 2.4. Compared with the nonlinear controller derived from the passive controller [5], the controller designed in this paper is simpler and free from the finding of the bounds of the states of the chaotic system (1). In Ref. [15], the authors proposed three controllers $u_1 = -k_1e_1(k_1 > -25\alpha - 8)$, $u_2 = -k_2e_2(k_2 > 29\alpha - 1 + (19 - 5\alpha - 0.5x_3)^2$ and $u_3 = -k_3e_3(k_3 > 0.25y_1^2 - (8+\alpha)/3)$ to realize synchronization of the unified chaotic systems. The controller gain is larger and they should also determine the bounds of chaotic states x_3 and y_1.

3. Simulation results

In this section, the fourth-order Runge–Kutta integration method is used to obtain the solutions of differential equations with step size 0.001. The initial states of the controlled Lorenz and Chen systems are $x(0) = 10$, $y(0) = -10$, $z(0) = 10$. Figs. 1 and 2 show that the Lorenz and Chen system can be stabilized to the origin $(0,0,0)$ with the linear control law designed in Section 2.2.

Choose the initial conditions of the master system (1): $x_1(0) = -1, x_2(0) = -1, x_3(0) = 1$, and of the slave system (11): $y_1(0) = 4, y_2(0) = -4, y_3(0) = 4$. When $\alpha = 0$ system (1) is the Lorenz chaotic system. When

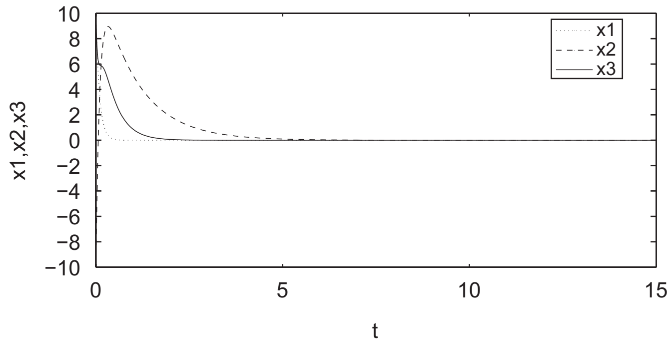

Fig. 1. The time response of the states for the controlled Lorenz system ($\alpha = 0$).

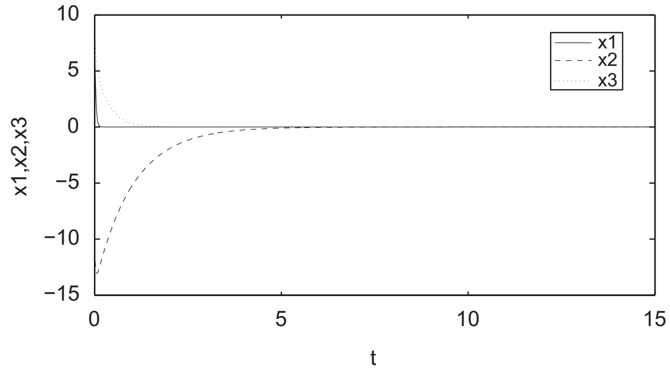

Fig. 2. The time response of the states for the controlled Chen system ($\alpha = 1$).

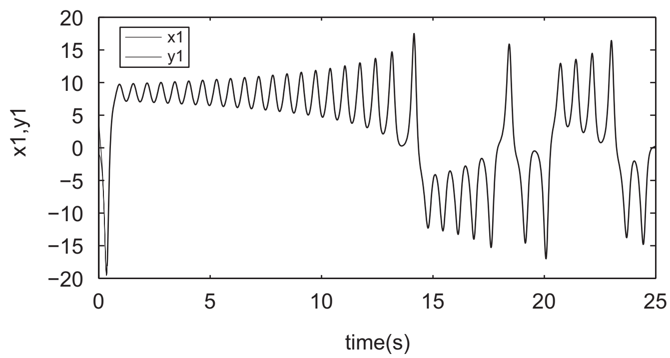

Fig. 3. Synchronization of two identical Lorenz systems x_1 and y_1 ($\alpha = 0$).

$\alpha = 0.8$ the system is a Lü chaotic system. The simulation results for synchronization of the Lorenz and Lü chaotic systems are shown in Figs. 3–6, respectively. Figs. 3–5 show the synchronization of the Lorenz chaotic system. Fig. 6 shows the time response of the error system for the synchronization of the Lü chaotic system. As expected, one can observe that the trajectories of the slave system asymptotically approach those of the master system as illustrated in Figs. 3–6.

4. Conclusion

Based on the stability theory of the cascade-connected systems, a novel method is developed to realize chaos control and synchronize the unified chaotic systems. It is obvious that the feedback given in this paper is very

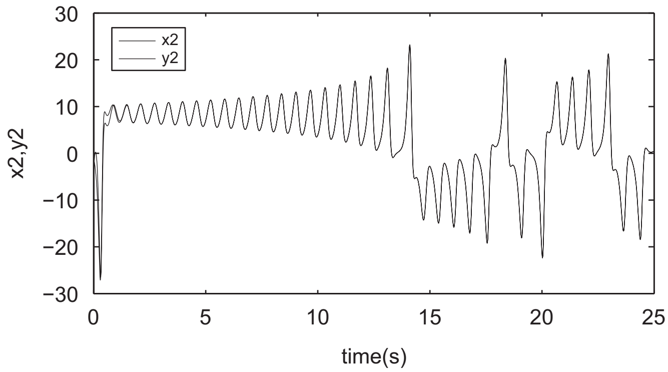

Fig. 4. Synchronization of two identical Lorenz systems x_2 and y_2 ($\alpha = 0$).

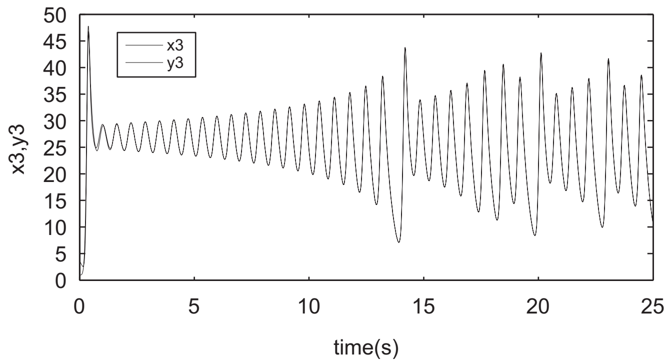

Fig. 5. Synchronization of two identical Lorenz systems x_3 and y_3 ($\alpha = 0$).

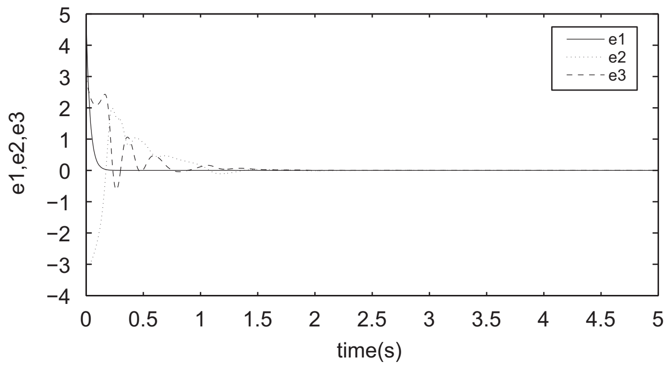

Fig. 6. Synchronization errors between master and slave Lü chaotic systems ($\alpha = 0.8$).

simple. Moreover, when $0 \leqslant \alpha < \frac{1}{29}$ only one linear controller is required to realize the chaotic control and synchronization for the unified chaotic systems. The effectiveness of this proposed synchronization method has been validated by numerical simulation results for Lorenz, Lü and Chen chaotic systems, respectively.

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